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SO(2, 1) coherent-state Green function for the Klein–Gordon Coulomb problem

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Abstract. We use the Perelomov SO(2, 1) coherent states to evaluate a Green function of a relativistic charged spinless particle in a Coulomb potential and determine its bound-state energy spectrum.

1. Introduction

The problem of a relativistic charged spinless particle in an external Coulomb field arises when we study, for example, the problem of a π-mesonic atom.

The quantum mechanical description of this problem is given by the Klein-Gordon equation [1], the solution of which describes the bound-state energy spectrum and transition rates.

An elegant solution for this problem is the algebraic one, generated by the SO(2, 1) dynamical symmetry of the problem. One possibility is to construct a Green function, in coordinate space, using differential operator realisation for the SO(2, 1) Lie algebra [2]. The use of dynamical symmetry algebras to find the spectrum or the Green functions is not new as may be seen from previous work [3] which treated different problems.

In this paper we explore another possibility, that is to construct a Green function for this problem using the SO(2, 1) Lie algebra coherent states as defined by Perelomov [4], the exponential Schwinger representation [5] for the resolvent operator and the Baker-Campbell-Hausdorff formulae [6]. This technique is similar to that of Gerry and Silverman [7], who construct Green functions over coherent states using path integrals. In our case we construct directly the Green function, suppressing path integral formulation, which is not necessary anyway.

This paper is organised as follows. In § 2 we construct the Schwinger representation for the resolvent operator for a relativistic charged spinless particle in an external Coulomb field. In § 3 we review the basic aspects of SO(2, 1) Lie algebra, its coherent states and derive two Baker-Campbell-Hausdorff formulae which will be used in later sections. In § 4 we construct, explicitly, the Green function over SO(2, 1) coherent states and find the energy spectrum from its poles and in § 5 we present the conclusions.

2. Statement of the problem

The quantum mechanical motion of a free relativistic spinless particle of mass m can be described by a Green function $G(x, x')$ which satisfies the free Klein-Gordon

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equation:

$$(\Box + m^2)G(x, x') = \delta^4(x - x') \tag{1}$$

where $\Box$ is the d'Alembertian operator. When a particle of charge e moves in a region where there exists a Coulomb potential, $V(r) = -Ze/r$, where Z is the atomic number, we must do a gauge-invariant substitution

$$\mathrm{i}\frac{\partial}{\partial t} \to \mathrm{i}\frac{\partial}{\partial t} - eV(r)$$

modifying the free Klein-Gordon equation to ($\alpha = e^2$)

$$\left[-\left(\mathrm{i}\frac{\partial}{\partial t} + \frac{Z\alpha}{r}\right)^2 - \nabla^2 + m^2\right]G(x, x') = \delta^4(x - x'). \tag{2}$$

Multiplying this equation by the radial coordinate r, one can write for $G(x, x')$ the Schwinger representation [5], in spherical polar coordinates as

$$G(x, x') = \mathrm{i}\int_0^\infty \mathrm{d}s \exp\left\{-\mathrm{i}sr\left[-\left(\mathrm{i}\frac{\partial}{\partial t} + \frac{Z\alpha}{r}\right)^2 + p_r^2 + \frac{L^2}{r^2} + m^2\right]\right\} r\delta^4(x - x') \tag{3}$$

where p_r is the radial momentum

$$p_r = -\frac{\mathrm{i}}{r}\left(\frac{\partial}{\partial r}\right) r \tag{4}$$

and L the usual angular momentum operator. Making the Fourier transform

$$G_E(x, x') = \int_{-\infty}^{+\infty} G(x, x')\mathrm{e}^{-\mathrm{i}Et}\,\mathrm{d}t \tag{5}$$

equation (3) becomes

$$G_E(x, x') = G_E r\delta^3(x - x') \tag{6}$$

where the resolvent operator G_E is given by

$$G_E = \mathrm{i}\int_0^\infty \mathrm{d}s \exp\left\{-\mathrm{i}sr\left[-\left(E + \frac{Z\alpha}{r}\right)^2 + p_r^2 + \frac{L^2}{r^2} + m^2\right]\right\} \tag{7}$$

which will be written, in following sections, in terms of SO(2, 1) generators.

3. SO(2, 1) Lie algebra

We now introduce the operators [8]

$$K_0 = \frac{1}{2m}\left(rp_r^2 + \frac{L^2 - Z\alpha}{r} + m^2 r\right) \tag{8a}$$

$$K_1 = \frac{1}{2m}\left(rp_r^2 + \frac{L^2 - Z\alpha}{r} - m^2 r\right) \tag{8b}$$

$$K_2 = rp_r \tag{8c}$$

which satisfy the canonical commutation relations for the SO(2, 1) Lie algebra

$$[K_0, K_1] = \mathrm{i}K_2 \tag{9a}$$

$$[K_1, K_2] = -\mathrm{i}K_0 \tag{9b}$$

$$[K_2, K_0] = \mathrm{i}K_1. \tag{9c}$$

In the $D^+(k)$ infinite discrete representation of the SO(2, 1) Lie algebra [8], the operator K_0 is diagonal for states $|k, n\rangle$:

$$K_0|k, n\rangle = (k+n)|k, n\rangle \tag{10}$$

and also the Casimir operator

$$\begin{aligned} C|k, n\rangle &= (K_0^2 - K_1^2 - K_2^2)|k, n\rangle \\ &= k(k-1)|k, n\rangle \end{aligned} \tag{11}$$

which in realisation (8) is

$$C = l(l+1) - Z\alpha \tag{12}$$

where $l(l+1)$ is the eigenvalue of the L^2 operator. Combining (11) and (12), we choose the Bargmann [9] index k as

$$k = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + l(l+1) - Z\alpha}. \tag{13}$$

The SO(2, 1) coherent states, according to Perelomov [4] are given by

$$|\zeta, k\rangle = (1-|\zeta|^2)^k \exp\{\zeta K_+\}|k, 0\rangle \tag{14}$$

or

$$|\zeta, k\rangle = (1-|\zeta|^2)^k \sum_{n=0}^{\infty} \left[\frac{\Gamma(2k+n)}{n!\Gamma(2k)}\right]^{1/2} \zeta^n |k, n\rangle \tag{15}$$

where K_+ is the ladder operator

$$K_+ = K_1 + \mathrm{i}K_2 \tag{16}$$

that creates the excited state $|k, n\rangle$, from the fundamental state $|k, 0\rangle$:

$$|k, n\rangle = \left[\frac{\Gamma(2k)}{n!\Gamma(2k+n)}\right]^{1/2} (K_+)^n |k, 0\rangle. \tag{17}$$

The overlap of two coherent states is given by

$$\langle \zeta', k|\zeta, k\rangle = (1-|\zeta|^2)^k (1-|\zeta'|^2)^k (1-\zeta'^*\zeta)^{-2k} \tag{18}$$

while a completeness relation can be established as

$$1 = \frac{2k-1}{\pi} \int d^2\zeta (1-|\zeta|^2)^{-2} |\zeta, k\rangle\langle\zeta, k|. \tag{19}$$

Another set of coherent states, that still obeys (18) and (19), called the physical coherent states, could be defined through a pseudo-rotation, generated by the non-compact operator K_2:

$$|\widetilde{\zeta, k}\rangle = \exp\{\mathrm{i}\theta K_2\}|\zeta, k\rangle \tag{20}$$

with an arbitrary angle θ, which will be conveniently chosen.

This transformation could be equally well applied to K_0 and K_1 operators, according to the BCH formulae [6]

$$e^{-i\theta K_2} K_0 \, e^{i\theta K_2} = K_0 \cosh\theta + K_1 \sinh\theta \tag{21a}$$

$$e^{-i\theta K_2} K_1 \, e^{i\theta K_2} = K_0 \sinh\theta + K_1 \cosh\theta \tag{21b}$$

so that a linear combination

$$\tilde{\Lambda} = aK_0 + bK_1 \tag{22}$$

submitted to this transformation is

$$\begin{aligned} \Lambda &= e^{-i\theta K_2} \tilde{\Lambda} \, e^{i\theta K_2} \\ &= K_0(a\cosh\theta + b\sinh\theta) + K_1(a\sinh\theta + b\cosh\theta) \end{aligned} \tag{23}$$

which is proportional to K_0 when we choose:

$$\theta = \ln\left[\frac{a-b}{a+b}\right]^{1/2} \tag{24}$$

leaving

$$\Lambda = cK_0 \tag{25}$$

where

$$c = [(a-b)(a+b)]^{1/2}. \tag{26}$$

4. The Green function defined over physical coherent states

Defining the Green function over physical coherent states are

$$G_E(\zeta, \zeta') = \sum_k \langle \widetilde{\zeta, k} | G_E | \widetilde{\zeta', k} \rangle \tag{27}$$

where the resolvent operator G_E, given by (7), could be written in terms of a linear combination $\tilde{\Lambda}$, (equation (22)) as

$$G_E = i \int_0^\infty ds \exp\{-is(\tilde{\Lambda} - 2EZ\alpha)\} \tag{28}$$

where the coefficients a and b of $\tilde{\Lambda}$, are identified as

$$a = \frac{2m^2 - E^2}{m} \qquad b = \frac{E^2}{m} \tag{29}$$

so the Green function (27) becomes

$$\begin{aligned} G_E(\zeta, \zeta') &= i \sum_k \int_0^\infty ds \langle \widetilde{\zeta, k} | \exp\{-is(\tilde{\Lambda} - 2EZ\alpha)\} | \widetilde{\zeta', k} \rangle \\ &= i \sum_k \int_0^\infty ds \langle \zeta, k | \exp\{-is(cK_0 - 2EZ\alpha)\} | \zeta', k \rangle \end{aligned} \tag{30}$$

where

$$c = 2\sqrt{m^2 - E^2} \tag{31}$$

according to (20) and (23)-(26). Using (10), (15) and (18) and making a Taylor expansion of the exponential of K_0, one may write

$$G_E(\zeta, \zeta') = \mathrm{i} \sum_k \int_0^\infty \mathrm{d}s \exp\{-\mathrm{i}s(2k\sqrt{m^2-E^2} - 2EZ\alpha)\} \times \left(\frac{(1-|\zeta|^2)(1-|\zeta'|^2)}{[1-\zeta'\zeta^* \exp\{-2\mathrm{i}s\sqrt{m^2-E^2}\}]^2} \right)^k \tag{32}$$

which is the Green function for the Klein-Gordon Coulomb problem in SO(2, 1) coherent-state space. To find the spectrum of this system we must take the trace of the resolvent operator, using (19):

$$\begin{aligned} \mathrm{Tr}\, G_E &= \sum_k \frac{2k-1}{\pi} \int \mathrm{d}^2\zeta (1-|\zeta|^2)^{-2} G_E(\zeta, \zeta) \\ &= \mathrm{i} \sum_k \int_0^\infty \mathrm{d}s \exp\{-2\mathrm{i}s(k\sqrt{m^2-E^2} - EZ\alpha)\}(2k-1) \int_0^1 \mathrm{d}|\zeta|\,|\zeta| \\ &\quad \times (1-|\zeta|^2)^{-2} \left(\frac{1-|\zeta|^2}{1-|\zeta|^2 \exp\{-2\mathrm{i}s\sqrt{m^2-E^2}\}} \right)^{2k} \\ &= \mathrm{i} \sum_k \int_0^\infty \mathrm{d}s \frac{\exp\{-2\mathrm{i}s(k\sqrt{m^2-E^2} - EZ\alpha)\}}{1-\exp\{-2\mathrm{i}s\sqrt{m^2-E^2}\}}. \end{aligned} \tag{33}$$

Expanding

$$[1-\exp\{-2\mathrm{i}s\sqrt{m^2-E^2}\}]^{-1} = \sum_{q=0}^\infty \exp\{-2\mathrm{i}qs\sqrt{m^2-E^2}\} \tag{34}$$

where we omit, for simplicity, a convenient convergence factor, we get

$$\mathrm{Tr}\, G_E = -\mathrm{i} \sum_k \sum_q [2EZ\alpha - 2(k+q)\sqrt{m^2-E^2}]^{-1} \tag{35}$$

which has poles at

$$E = m\left[1 + \frac{Z^2\alpha^2}{(k+q)^2}\right]^{-1/2} \tag{36}$$

which fit the known spectrum [1] for a relativistic bound spinless particle in a Coulomb field.

5. Conclusions

In this paper we show the possibility of describing a relativistic system through generalised coherent states, constructing its Green function and finding the corresponding energy spectrum.

This technique could be equally well applied to other systems with SO(2, 1) dynamical symmetry like the non-relativistic Coulomb potential, the three-dimensional harmonic oscillator and the Morse potential.

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